On the reduction modulo p of representations of a quaternion division algebra over a p-adic field

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Abstract

The p-adic Langlands correspondence and the mod p Langlands correspondence for $GL_2(\mathbb{Q}_p)$ are known to be compatible with the reduction modulo p in many cases. We examine whether there exists a similar compatibility for the composite of the local Langlands correspondence and the local Jacquet-Langlands correspondence. The simplest case has already been considered by Vignéras. We deal with more cases.

1. Introduction

Recently two kinds of analogues of the local Langlands correspondence for GL_n have been pursued extensively; the p-adic Langlands correspondence and the mod p Langlands correspondence. For $GL_2(\mathbb{Q}_p)$, two correspondences have been established ([2]). Among many other properties they are compatible with the reduction modulo p.

Theorem 1. (*Berger [1]*)
R: two-dimensional absolutely irreducible p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$
Π: irreducible unitary Banach representation of $GL_2(\mathbb{Q}_p)$ which corresponds to R via the p-adic Langlands correspondence
$\overline{R}^{ss}$: (the semi-simplification of) the reduction modulo p of R
$\overline{\Pi}^{ss}$: (the semi-simplification of) the reduction modulo p of Π
Suppose that R is trianguline.
Then $\overline{R}^{ss}$ corresponds to $\overline{\Pi}^{ss}$ via the mod p Langlands correspondence.

Our objective here is to look for a similar compatibility in a related situation with GL_2 replaced by the multiplicative group of a quaternion division algebra over a p-adic field.

2. Our setting

F: non-Archimedean local field with a finite residue field with $q = p^d$ elements
D: quaternion division algebra over F
U_D^i: (higher) unit group of D
$\mathcal{W}_F$: Weil group of F
Let us fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$. The local Langlands correspondence and the local Jacquet-Langlands correspondence (when suitably restricted) defines a canonical bijection between the following sets

$$\{\text{two-dimensional irreducible smooth representations of } \mathcal{W}_F/\overline{\mathbb{Q}}_p\}/\cong$$
$$\overset{\mathrm{LL}}{\to} \{\text{supercuspidal representations of } GL_2(F)/\overline{\mathbb{Q}}_p\}/\cong$$
$$\overset{\mathrm{JL}}{\to} \{\text{irreducible admissible representations of } D^\times/\overline{\mathbb{Q}}_p \text{ which are not one-dimensional}\}/\cong$$

In particular, we have a natural correspondence between representations of $\mathcal{W}_F$ and of $D^\times$ over $\overline{\mathbb{Q}}_p$.
On the other hand, classifying representations, we obtain a natural correspondence ("the mod p correspondence") between mod p representations of $\mathcal{W}_F$ and of $D^\times$, more precisely, between

$$\{\text{two-dimensional irreducible smooth representations of } \mathcal{W}_F/\overline{\mathbb{F}}_p\}/\cong$$

and

$$\{\text{irred. smooth representations of } D^\times \text{ with a central character } /\overline{\mathbb{F}}_p, \text{ not one-dimensional}\}/\cong$$

Remark 2.

1. *Every irreducible mod p representation of $D^\times$ is inflated from that of $D^\times/U_D^1$ and similarly every irreducible mod p representation of $\mathcal{W}_F$ is inflated from that of $\mathcal{W}_F/\mathcal{P}_F$, where $\mathcal{P}_F$ is the wild inertia subgroup of $\mathcal{W}_F$. Thanks to these facts the classification of the representations is not difficult and moreover the groups essentially involved are isomorphic.*
2. *It also follows from the classification that every irreducible smooth representation of $D^\times$ with a central character is either one-dimensional or two-dimensional.*

3. Main result

R: p-integral two-dimensional irreducible representation of $\mathcal{W}_F$
Π: p-integral irreducible admissible representation of $D^\times$ which corresponds to R via the composite of the local Langlands correspondence and the local Jacquet-Langlands correspondence
$\overline{R}^{ss}$: (the semi-simplification of) the reduction modulo p of R
$\overline{\Pi}^{ss}$: (the semi-simplification of) the reduction modulo p of Π
Taking Remark 2 into account, we ask the following questions.

> Is $\overline{\Pi}^{ss}$ isotypic?
> If so, is its irreducible factor determined by $\overline{R}^{ss}$ by the mod p correspondence?

Theorem 3.

1. (*Vignéras [6]*)
   Suppose that Π is trivial on U_D^1 (or equivalently R is tamely ramified).
   Then $\overline{R}^{ss}$ and $\overline{\Pi}^{ss}$ are irreducible and they correspond via the mod p correspondence.
2. (*T.*)
   Suppose that p is odd and that Π is not trivial on U_D^1.
   Then $\overline{\Pi}^{ss}$ is not irreducible nor isotypic.

Remark 4.

1. *There exist only a finite number of (isomorphism classes of) mod p representations of $D^\times$ with the given central character. Under the assumption of 2., all irreducible representations with the appropriate central character occur in $\overline{\Pi}^{ss}$ if the dimension of Π is sufficiently large.*
2. *In fact, we determined the irreducible decomposition of $\overline{\Pi}^{ss}$ for all p-integral irreducible admissible representation Π of $D^\times$ if $p \neq 2$ and for all such representations which are tame in some sense if $p = 2$.*
3. *The computation of the irreducible decomposition of $\overline{\Pi}^{ss}$ makes use of the expression of Π as an induced representation ([4]).*

4. Some remarks

1. In a way, it may be natural that the mod p correspondence and the composite of the local Langlands correspondence and the local Jacquet-Langlands correspondence are compatible with the reduction modulo p only in the case of Theorem 3.1:
   - Irreducible mod p representations of $D^\times$ are automatically trivial on U_D^1 and irreducible mod p representations of $\mathcal{W}_F$ are automatically tamely ramified. Therefore, it is no wonder if the compatibility holds only for the representations in characteristic zero satisfying the corresponding conditions.
   - Two-dimensional potentially semi-stable trianguline representations are known to be related to two-dimensional reducible Weil-Deligne representations (via Fontaine's functor D_{pst}), which in turn correspond to non-supercuspidal representations—the class of representations of $GL_2(\mathbb{Q}_p)$ not appearing in the local Jacquet-Langlands correspondence([5]).
2. Still, if $\overline{R}^{ss}$ is irreducible, the image π of $\overline{R}^{ss}$ under the mod p correspondence does occur in $\overline{\Pi}^{ss}$ in every case. We may at least ask if there exists any way to single out π among other irreducible factors occurring in $\overline{\Pi}^{ss}$.
   (a) If $\dim \Pi = 2q^e$ for some e, then π can be characterized as *the only two-dimensional irreducible factor with the multiplicity distinct from that of any other two-dimensional factors*.
   (b) If $\dim \Pi = (q+1)q^e$ for some e, then all the two-dimensional irreducible factors have the same multiplicities and π can be characterized as *the only two-dimensional irreducible factor π such that $\pi(u^2)$ is a scalar operator for any $u \in U_D$*.

References

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